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ON CONTINUITY OF THE UTILITY FUNCTION IN INTERTEMPORAL ALLOCATION MODELS: AN EXAMPLE

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A standard model of intertemporal allocation (described by a technology set, and a welfare function defined on consumption) can be reduced to one described by a technology set, and a utility function defined on this set. We present an example to show that even when the welfare function is concave, monotone and continuous, the utility function can be discontinuous. We also provide sufficient conditions on the technology set and the welfare function under which the utility function is continuous. Our results indicate that the common practice of *assuming* continuity of the utility function is more restrictive than might be apparent.

1. INTRODUCTION

A well-known model of optimal intertemporal allocation is described by a technology set of input-output pairs, Ω, and a welfare function, w, defined on (nonnegative) consumption bundles. (Strictly speaking, one also specifies a discount factor to represent intertemporal preferences, but this will not be of direct concern to us here.) It is standard to assume regarding Ω that inaction is possible, and free production is impossible; production possibilities are closed and convex; free disposal is allowed, and it is impossible to sustain "large" input levels. Similarly, it is standard to assume regarding w that it is concave and upper semicontinuous.

Given this framework, we can associate with any input pair (x, z) a *feasible output set*, $g(x, z)$, which is the set of outputs, y, producible from x, such that $y \geqslant z$. That is, given an input level for this period, x, and an input level for next period, z (which can be technologically attained in one period from x), $g(x, z)$ is the set of outputs which allow one to reach the input level, z, after consuming a nonnegative amount $(y\text{-}z)$. Under free disposal the set of feasible input pairs is of course exactly Ω, which is hence the domain of g. (In the sequel, Ω shall interchangeably refer to the set of feasible input-input and input-output pairs and the correct interpretation shall be clear from the context.) Then, one can associate with a pair (x, z) on Ω, a *utility level*, $u(x, z)$, defined as the maximum welfare obtainable among such consumption bundles $(y\text{-}z)$, where y is in the feasible output set. (For precise definitions of these concepts, see Section 2.)

Thus, the "consumption model" has been "reduced" to one which can adequately be described by a technology set, Ω (satisfying the standard assumptions stated above), and a utility function, u, defined on Ω. This "reduced" framework

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has been studied by a number of authors (see Gale 1967; McKenzie 1968, 1976, 1982; Brock 1970; Sutherland 1970; Flynn 1980) in developing the major results of optimal intertemporal allocation theory. For this purpose the utility function is *assumed* to be concave, non-decreasing in this period's input, non-increasing in next period's input, and *continuous* on Ω.

In order to treat the consumption model as a special case of these studies, then, we have to check whether the utility function, as defined above by us, can be *shown* to satisfy these properties. We establish (in Section 2) that the utility function is concave, non-decreasing in this period's input, non-increasing in next period's input, and *upper semicontinuous* on Ω.

The important question that remains to be answered then is whether or not the utility function is continuous as well, on Ω. We provide an example in Section 3 of a technology set, Ω (satisfying all of the above-stated assumptions), and a welfare function, w (which is concave, continuous and monotone in consumption), such that the utility function, u, generated by them fails to be continuous. Since continuity of u is used at several crucial points in the above-mentioned studies, the example demonstrates that it is important to develop the major themes of optimal intertemporal allocation theory (possibly by using some alternative methods) without the continuity assumption on u. Some of the contributions in this direction include Peleg (1973), Khan and Mitra (1986), Dutta and Mitra (1985), and McKenzie (1986).

The example of Section 3 turns out to be far from simple to construct, and points to the possibility that for some broad classes of technology sets or welfare functions, the utility function may be shown to be continuous. In Section 4, we confirm this by establishing the continuity of the utility function when either (i) the technology set is "locally simplicial," or (ii) the welfare function is "indecomposible."

2. PRELIMINARIES

This section is organized in the following way. Subsection 2a introduces most of the notation and definitions used in the paper. The second subsection 2b describes a familiar model of intertemporal allocation theory in terms of a technology set, Ω, and a welfare function, w, where welfare is derived from consumption of goods. The model is then "reduced" to one in which *utility* is derived from this period's and next period's input levels. We show in 2c, that the utility function, so defined, has some useful properties which can be derived from the standard assumptions on the technology set, Ω, and the welfare function, w.

2a. *Notation and Definitions.* Let R^m be an m-dimensional real space, with Euclidean norm, $\|\cdot\|$. For x, y in R^m, $x \geqslant y$ means $x_j \geqslant y_j$ for $j = 1, \ldots, m$; $x > y$ means $x \geqslant y$ and $x \neq y$; $x \gg y$ means $x_j > y_j$ for $j = 1, \ldots, m$. The set $\{x \in R^m: x \geqslant 0\}$ is denoted by R^m_+.

Let A, B be subsets of R^l and R^m respectively. Let $a^o \in A$. (i) A correspondence G: $A \to B$ is *lower hemicontinuous at* a^o if "$a^n \to a^o$ and $b^o \in G(a^o)$" imply that there is an integer $\bar{n}$, and a sequence $b^n \in G(a^n)$ for $n \geqslant \bar{n}$, such that

$\lim_{n\to\infty} b^n = b^o$. (ii) G is *upper hemicontinuous at* a^o if $G(a^o)$ is nonempty and compact, and "$a^n \to a^o$, $b^n \in G(a^n)$ for all n" imply that there is a converging subsequence of (b^n), whose limit belongs to $G(a^o)$. (iii) The correspondence G: $A \to B$ is *continuous* at a^o if it is both upper and lower hemicontinuous at a^o.

The correspondence G: $A \to B$ is lower (upper) hemicontinuous on A, if it is lower (upper) hemicontinuous at each point of A. It is continuous on A, if it is both upper and lower hemicontinuous on A.

A function, f: $A \to R$ is *upper semicontinuous* at $a^o \in A$, if "$a^n \in A$, and $a^n \to a^o$" imply "$\lim_{n\to\infty} \sup f(a^n) \leq f(a^o)$." It is upper semicontinuous on A if it is upper semicontinuous at each point of A. It is *lower semicontinuous* at $a^o \in A$ [on A] if $(-f)$ is upper semicontinuous at $a^o \in A$ [on A]. It is *continuous* at $a^o \in A$ [on A] if f is both upper and lower semicontinuous at $a^o \in A$ [on A].

2b. *The Model.* The framework is described by (Ω, w), where Ω, a subset of $R^m_+ \times R^m_+$, is the *technology set* and w: $R^m_+ \to R$ is the *welfare function*. The following assumptions on Ω and w are fairly standard:

(A.1) (i) $(0, 0) \in \Omega$, (ii) $(0, y) \in \Omega$ implies $y = 0$.
(A.2) Ω is a closed subset of $R^m_+ \times R^m_+$.
(A.3) Ω is a convex set.
(A.4) If $(x, y) \in \Omega$, and $x' \geq x$, and $0 \leq y' \leq y$, then $(x', y') \in \Omega$.
(A.5) There is a positive number β, such that "$(x, y) \in \Omega$ and $\|x\| > \beta$" imply $\|y\| \leq \|x\|$.
(A.6) w is a concave function on R^m_+.
(A.7) w is upper semicontinuous on R^m_+.

For $(x, z) \in \Omega$ (here z is "next period's input"), we can define a *feasible output correspondence* by

$$g(x, z) = \{y \in R^m_+ : (x, y) \in \Omega, \quad \text{and} \quad y \geq z\} \tag{2.1}$$

Thus, $g(x, z)$ is the set of outputs producible from input x which makes it possible to have an input vector of z (in the next period), after some nonnegative consumption $((y - z) \geq 0)$.

Note that g is a correspondence from Ω to R^m_+; it is non-empty valued (since $z \in g(x, z)$) and closed valued (since Ω is closed). Furthermore, it is well-known that if $(x, y) \in \Omega$ then $\|y\| \leq \max\,[\|x\|, \beta] \equiv B(x)$, where β is given by (A.3). Thus g is compact valued.

For $(x, z) \in \Omega$ (again, z is "next period's input"), we can define a *utility function* by

$$u(x, z) \equiv \max\, w(y - z) \quad \text{subject to} \quad y \in g(x, z). \tag{2.2}$$

Since w is upper semicontinuous (and g is nonempty and compact valued) so u is well-defined on Ω. It is the maximum welfare obtainable, if this period's input is x, and next period's input is z, through suitable production and consumption in the next period.

2c. *Upper Semicontinuity and Concavity of the Utility Function.* We will show in the next section that the utility function is not, in general, continuous. However, one can demonstrate that the utility function is upper semicontinuous and concave.

THEOREM 2.1. *Suppose* (Ω, w) *satisfy* (A.1) *through* (A.7). *Then* (i) *the feasible output correspondence, g, is upper hemicontinuous on* Ω, *and* (ii) *the utility function, u, is upper semicontinuous and concave on* Ω. *Also, if* $(x, z) \in \Omega$, *and* $x' \geqslant x$, *and* $0 \leqslant z' \leqslant z$, *then* $u(x', z') \geqslant u(x, z)$.

PROOF. Let $(x^o, z^o) \in \Omega$. Define, as usual, $B(x) \equiv \max(\beta, \|x\|)$; then for $(x, z) \in \Omega$, $\|z\| \leqslant B(x)$ (see Khan and Mitra 1986). Next, define $N(x^o, z^o) = \{(x, z) \in \Omega: \|x\| \leqslant B(x^o) + 1, \|z\| \leqslant B(x^o) + 1\}$. If $(x, z) \in N(x^o, z^o)$, and $y \in g(x, z)$, then $\|y\| \leqslant B(x) = \max(\beta, \|x\|) \leqslant \max(\beta, B(x^o) + 1) = B(x^o) + 1$.

Since g is a non-empty and compact valued correspondence, we can check its upper hemicontinuity at (x^o, z^o) as follows. Let $(x^n, z^n) \to (x^o, z^o)$, and $y^n \in g(x^n, z^n)$. Clearly, $((x^n, z^n) \in N(x^o, z^o)$ for all n beyond some $\bar{n}$. Then since $\|y^n\| \leqslant B(x^o) + 1$, as checked above, so there is a convergent subsequence $(y^{n'})$ converging to y^o. Since $(x^{n'}, y^{n'}) \in \Omega$ and $y^{n'} \geqslant z^{n'}$ and $(x^{n'}, y^{n'}, z^{n'}) \to (x^o, y^o, z^o)$, so $(x^o, y^o) \in \Omega$ and $y^o \geqslant z^o$ (since Ω is closed). So $y^o \in g(x^o, z^o)$, and g is upper hemicontinuous at (x^o, z^o). Since (x^o, z^o) was arbitrary in Ω, so g is upper hemicontinuous on Ω. Now, by adapting the argument of Berge (1963, p. 116), u is upper semicontinuous on Ω.

From the fact that the graph of $g \equiv \{(x, z, y) \in R_+^{3m}: y \in g(x, z)\}$ is convex and w is concave, it easily follows that u is concave on Ω.

Since $x' \geqslant x$, and $0 \leqslant z' \leqslant z$ implies that $g(x', z') \supset g(x, z)$, it easily follows that $u(x', z') \geqslant u(x, z)$.

REMARK. Peleg (1973) establishes a result analogous to Theorem 2.1. However, his assumptions on the technology set and welfare function differ somewhat from ours (in particular, Ω is compact in his framework, but not in ours). We have therefore, presented the result with the proof to keep the exposition self-contained.

3. DISCONTINUITY OF THE UTILITY FUNCTION: AN EXAMPLE

Let us now turn to the following question: Is the reduced utility function u continuous on Ω? A number of authors (Gale 1967; McKenzie 1968, 1976, 1982; Brock 1970; Sutherland 1970; Flynn 1980; among others) have used the input model (Ω, u) but under the *assumption that the utility function u is continuous on the technology set* Ω. If the input model is to be seen as the more general framework of which the (above) consumption model is a special case, one needs to *show* that the reduced utility function is indeed continuous on Ω.

Note that since R_+^m is boundedly polyhedral (see Gale, Klee and Rockafellar 1968, p. 869), so (A.6) and (A.7) imply

(A.8) w is continuous on R_+^m.

Given this, it seems reasonable to try to use the Maximum Theorem of Berge (1963) to establish continuity of u. The answer then clearly hinges on the *continuity of the feasible output correspondence*, g, defined in equation (2.1). And since we already know from Theorem 2.1 that g is upper hemicontinuous on Ω, so the critical issue is the *lower hemicontinuity* of g on Ω.

The purpose of this section is to present an example of a technology set, Ω, and a welfare function, w, such that (a) Ω satisfies (A.1) through (A.5), and (b) w satisfies (A.6), (A.8) (and is monotone increasing in consumption) such that the feasible output correspondence, g, fails to be lower hemicontinuous, and furthermore the utility function, u, defined by equation (2.2), fails to be continuous.

The example we present below is not a simple one; nor is it easy to construct. To see the difficulties, note right away that since u is concave on Ω (Theorem 2.1), it is continuous in the interior of Ω, so discontinuities, if any, can arise only at the boundary. Since u is also upper semicontinuous on Ω, behavior even at the boundary is considerably restricted. Also, note that the feasible output correspondence, g, is the intersection of two correspondences both of which are lower hemicontinuous. In general, the intersection of two lower hemicontinuous correspondences is *not* lower hemicontinuous (which "explains" why we *can* construct an example); however, it is easy to check that in the one-good model ($m = 1$), the intersection of these correspondences is indeed lower hemicontinuous (which "explains" why we present a two-good example).

3a. *Description of the Technology Set.* Define a technology set Ω in $R^2_+ \times R^2_+$ in the following way. Denoting the vector (1, 1) by e, we have $ex = (x_1 + x_2)$ for every $x = (x_1, x_2)$ in R^2_+. For $x = (x_1, x_2) \in R^2_+$, define $A(x) = \{y = (y_1, y_2)$ in R^2_+: $(y_1, y_2) \leq (\sqrt{ex}, 0)$ if $ex < 1$; $(y_1, y_2) \leq (1, 1)$ if $ex \geq 1\}$. The *graph* of A is the set $\{(x, y)$ in $R^2_+ \times R^2_+$: $y \in A(x)\}$, and we denote it by gr$A$. Now, define the technology set to be the *convex hull* of gr$A$. That is, $\Omega \equiv$ con [gr$A$].

3b. *A Characterization of the Technology Set.*

$$\text{Let} \qquad C_1 \equiv \{(x, y) \text{ in } R^2_+ \times R^2 \colon x_1 + x_2 \leq 1\}$$

$$C_2 \equiv \{(x, y) \text{ in } R^2_+ \times R^2 \colon x_1 + x_2 \geq 1\}.$$

Clearly, con (gr$A$) = [con (gr$A$) $\cap$ C_1] $\cup$ [con (gr$A$) $\cap$ C_2]. It is straightforward to check that

$$\text{(3.1)} \qquad \text{con (gr}A) \cap C_2 = \{(x, y) \in R^2_+ \times R^2_+ \colon x_1 + x_2 \geq 1, y \leq e\}$$

$$= \text{con [gr}A \cap C_2].$$

We shall now show that

$$\text{(3.2)} \qquad \text{con (gr}A) \cap C_1 = \text{con [gr}A \cap C_1].$$

By Caratheodory's Theorem $(x, y) \in$ con (gr$A$) $\cap$ C_1 implies that

$$(x, y) = \sum_{i=1}^{5} \lambda^i (x^i, y^i),\ \lambda^i \geq 0,\ \sum_{i=1}^{5} \lambda^i = 1,\ (x^i, y^i) \in \text{gr}A,\ ex \leq 1.$$

Without loss of generality suppose $\lambda_i > 0$ for all i and for $i = 1, \ldots, k$, $ex^i < 1$. [If there are no such x^i, then it must be that $ex^i = 1$, $i = 1, \ldots, 5$ (and $ex = 1$)]. Denote, $\bar{x}_1 = \Sigma_{i=1}^{k} \lambda_i / \lambda_1 + \cdots + \lambda_k x_i$. Then $e\bar{x}_1 < 1$. Similarly define $\bar{y}_1$. Let us now define, $\bar{x}_2 = \Sigma_{i=k+1}^{5} \lambda_i / \lambda_{k+1} + \cdots + \lambda_5 x_i$ and $\bar{y}_2$ similarly. Again $e\bar{x}_2 \geq 1$. From the above we have

$$(x, y) = \mu_1(\bar{x}_1, \bar{y}_1) + \mu_2(\bar{x}_2, \bar{y}_2),$$

$\mu_1 = \Sigma_{i=1}^{k} \lambda_i$, $\mu_2 = \Sigma_{i=k+1}^{5} \lambda_i$. Hence, $\mu_1 + \mu_2 = 1$, $\mu_i \geq 0$, $i = 1, 2$. Further it is easy to check that

$$\bar{y}_1 = \sum_{i=1}^{k} \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_k} y^i \leq \sum_{i=1}^{k} \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_k} (\sqrt{ex^i}, 0) < (\sqrt{e(\bar{x}_1)}, 0)$$

So, $\bar{y}_1 \in A(\bar{x}_1)$. Also $\bar{y}_2 \leq e$ and hence, $\bar{y}_2 \in A(\bar{x}_2)$. Let us now define $\bar{x}_3$ as the intersection of the line-segment $\bar{x}_1$ to $\bar{x}_2$ with the hyperplane $ex = 1$. Clearly,

$$x = \theta_1 \bar{x}_1 + \theta_2 \bar{x}_2,$$

where $0 \leq \theta_1 \leq \mu_1$, $\theta_2 \geq \mu_2$ and $\theta_1 + \theta_2 = 1$. If $\bar{y}_3 = [(\mu_1 - \theta_1)/\theta_2]\,\bar{y}_1 + (\mu_2/\theta_2)\,\bar{y}_2$, then it is easy to check that $\bar{y}_3 \in \text{gr}(\bar{x}_3)$ and

$$y = \theta_1 \bar{y}_1 + \theta_2 \bar{y}_3.$$

All of the above establishes that, [con (gr$A$) $\cap$ C_1] $\subset$ con [(gr$A$) $\cap$ C_1]. The converse inclusion holds trivially and hence (3.2) is proved. Collecting (3.1) and (3.2) we have

(3.3) $$\Omega = \text{con}\,(\text{gr}A) = \text{con}\,[\text{gr}A \cap C_1] \cup \text{con}\,[\text{gr}A \cap C_2].$$

3c. *The Technology Set Satisfies* (A.1) *through* (A.5).

(1) (i) Since $0 \in A(0)$, so $(0, 0) \in \text{gr}A$ and hence is in con (gr$A$).
(ii) It is clear that if $0 = \Sigma_{i=1}^{5} \lambda^i x^i$, then $\lambda^i > 0$ implies $x^i = 0$. From this and the fact that $y^i \in \text{gr}A(0)$ implies $y^i = 0$, (A.1) (ii) follows.

(2) From (3.3) we can see that Ω is the union of two sets. The first is the convex hull of a compact set and hence is closed. The second is a closed set, as can be seen from (3.1). Hence, Ω is closed.

(3) Ω is convex by construction.

(4) It is easy to verify that if $(x, y) \in \text{gr}A$, and $\bar{x} \geq x$, and $0 \leq \bar{y} \leq y$, then $(\bar{x}, \bar{y}) \in \text{gr}A$. It remains to check that this (free-disposal) property is preserved by the convex-hull operation.

To this end, let $(x, y) \in$ con gr$A$, and $\bar{x} \geq x$, $0 \leq \bar{y} \leq y$. Define for $j = 1, 2$, $\mu_j = (\bar{y}_j / y_j)$ if $y_j > 0$, $\mu_j = 0$ if $y_j = 0$; then $0 \leq \mu_j \leq 1$, for $j = 1, 2$. Define the matrix

$$\mu = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$$

then $\bar{y} = \mu y$.

Since $(x, y) \in \Omega$, so by Carathedory's theorem, there exist $(x^i, y^i) \in \text{gr}A$, and λ^i $(i = 1, \ldots, 5)$, such that $\lambda^i > 0$, $\Sigma_{i=1}^5 \lambda^i = 1$, and $(x, y) = \Sigma_{i=1}^5 \lambda^i(x^i, y^i)$. Now, for $i = 1, \ldots, 5$, define $\bar{x}^i = x^i + (\bar{x} - x)$, and $\bar{y}^i = \mu y^j$. Then $\bar{x}^i \geqslant x^i$, and $0 \leqslant \bar{y}^i \leqslant y^i$, so $(\bar{x}^i, \bar{y}^i) \in \text{gr}A$ for $i = 1, \ldots, 5$. Also, $\Sigma_{i=1}^5 \lambda^i \bar{x}^i = \Sigma_{i=1}^5 \lambda^i x^i + (\bar{x} - x) \Sigma_{i=1}^5 \lambda^i = x + (\bar{x} - x) = \bar{x}$; and, $\Sigma_{i=1}^5 \lambda^i \bar{y}^i = \Sigma_{i=1}^5 \lambda^i \mu y^i = \mu \Sigma_{i=1}^5 \lambda^i y^i = \mu y = \bar{y}$. Hence $(\bar{x}, \bar{y}) \in \text{con}(\text{gr}A) = \Omega$.

(5) Note that for all $(x, y) \in \text{gr}A$, $0 \leqslant y \leqslant e$. Hence, for all $(x, y) \in \Omega$, $0 \leqslant y \leqslant e$. Pick $\beta = 2$. Then for $(x, y) \in \Omega$, $\|x\| > \beta$, we have $ex \geqslant \|x\| > 2$, while $\|y\| \leqslant ey \leqslant 2$, so that $\|y\| < \|x\|$.

3d. *A Property of the Technology Set.* For the purpose of our example the following property of the technology set will be crucial. *Let* $x = (x_1, x_2)$, $ex < 1$, $z = (\sqrt{ex}, 0)$. *Then* $(x, y) \in \Omega$, $y \geqslant z$ *implies* $z = y$. Note that (x, z) is in $\text{gr}A$, and so is in Ω. The assertion is interesting for it implies that (x, z) *cannot* be expressed as a convex combination of other points in the graph of A. Suppose to the contrary that $(x, y) \in \Omega$, $y \geqslant z$ and $(x, y) = \Sigma_{i=1}^5 \lambda^i(x^i, y^i)$, where without loss of generality, $\lambda^i > 0$, $\Sigma_{i=1}^5 \lambda^i = 1$ and $(x^i, y^i) \in (\text{gr}A \cap C_1)$ (using (3.3)) *and* not all $x^i = x$. Then,

$$y_1^i \leqslant \sqrt{ex^i} \quad \text{and so} \quad \sum_{i=1}^5 \lambda^i y_1^i \leqslant \sum_{i=1}^5 \lambda^i \sqrt{ex^i} < \sqrt{\Sigma \lambda^i (ex^i)} = \sqrt{ex} \tag{3.4}$$

Clearly (3.4) yields a contradiction and the assertion is proved.

3e. *The Feasible Output Correspondence is not Continuous.* We shall now show that the feasible output correspondence (see (2.1)) is not lower hemicontinuous.

Define $x^o = z^o = (1, 0)$. Define $x^n = (n^2/(n + 1)^2, 0)$ and $z^n = (\sqrt{ex^n}, 0)$. Clearly, $(x^n, z^n) \in \Omega$ for $n \geqslant 1$ and $(x^n, z^n) \to (x^o, z^o)$, as $n \to \infty$. Also, $y^o = (1, 1)$ is in $A(x^o)$ and hence (x^o, y^o) is in Ω. Clearly, $y^o \geqslant z^o$ and hence $y^o \in g(x^o, z^o)$. By the property proved in 3*d*, $y^n \in g(x^n, z^n)$ implies $y^n = z^n$. Thus for every sequence $\{y^n\}$, such that $y^n \in g(x^n, z^n)$, we have $\|y^n - y^o\| \geqslant 1$ for $n \geqslant 1$. Hence, g is not lower hemicontinuous at (x^o, z^o).

3f. *The Utility Function is not Continuous.* Define a welfare function, w: $R_+^2 \to R$ as follows. For $c = (c_1, c_2)$ in R_+^2, $w(c) = c_1 + c_2$. Then w satisfies (A.6), (A.8) and is monotone increasing in (c_1, c_2). For $(x, z) \in \Omega$, the utility function, u: $\Omega \to R$ is defined by:

$$u(x, z) = \max_{y \in g(x, z)} [(y_1 - z_1) + (y_2 - z_2)].$$

Note that for each $(x, z) \in \Omega$, the set of $y \in g(x, z)$ is a non-empty, compact set, and $e(y - z)$ is a continuous function of y. Hence $u(x, z)$ is well-defined, for each $(x, z) \in \Omega$.

Let (x^n, z^n) and (x^o, z^o) be defined as in Section 3e. Then since $y^n \in g(x^n, z^n)$ implies $z^n = y^n$, so $u(x^n, z^n) = 0$ for $n \geqslant 1$. However, $y^o \equiv (1, 1)$ is in $g(x^o, z^o)$, and so $u(x^o, z^o) \geqslant (y_2^o - z_2^o) = 1$. Thus, although (x^n, z^n), (x^o, z^o) are in Ω, and $(x^n, z^n) \to (x^o, z^o)$ as $n \to \infty$, we have $[u(x^o, z^o) - u(x^n, z^n)] \geqslant 1$ for all $n \geqslant 1$. So, u is not lower semicontinuous at (x^o, z^o).

REMARK. Peleg (1973) asserts that, in a framework in which u is derived from w and Ω by (2.2), u is not continuous even if w is continuous. We note that, in his framework, Ω is only restricted to be compact, convex and productive (there is (x, y) in Ω with $y \gg x$). An example displaying the lack of lower hemicontinuity of g, and a lack of lower semicontinuity of u, is therefore easier to construct in his framework than in ours. See, in this connection, the simple example discussed in Dutta and Mitra (1985, p. 4).

4. CONTINUITY OF THE UTILITY FUNCTION: TWO RESULTS

The example of the previous section shows that the utility function is not, in general, continuous. However, the difficulties of constructing such an example also point to the possibility that for some broad classes of technology sets or welfare functions, the utility function would in fact be continuous. In this section, we briefly discuss two such cases. One is the case where the technology set (besides satisfying (A.1) through (A.5)) is "locally simplicial," and the welfare function satisfies (A.6), (A.7). The other is the case where the technology set satisfies (A.1) through (A.5), and the welfare function (besides satisfying (A.6), (A.7)) is assumed to be "indecomposible" (which means that every good is essential for a welfare level higher than the welfare level from zero consumption).

Let A be a subset of R^m with $0 \in A$. A function $f: A \to R$ is called *indecomposible* if "$a \in A$ and $f(a) > f(0)$" imply "$a \gg 0$." We shall not define locally simplicial sets formally (see Rockafellar 1970; and Dutta and Mitra 1985). It suffices to note that they include simplices, polytopes and polyhedral convex sets.

THEOREM 4.1. *Suppose Ω satisfies* (A.1) *through* (A.5), *and w satisfies* (A.6), (A.7). *Further, suppose Ω is locally simplicial. Then, the utility function, u, is continuous and concave on Ω.*

PROOF. From Theorem 2.1, we know that u is upper semicontinuous and concave on Ω. Since Ω is locally simplicial and u is concave on Ω, so u is lower semicontinuous on Ω as well, by adapting the argument of Gale, Klee and Rockafellar (1968, p. 868) or Rockafellar (1970, p. 84). Hence, u is continuous and concave on Ω.

THEOREM 4.2. *Suppose Ω satisfies* (A.1) *through* (A.5), *and w satisfies* (A.6), (A.7). *Further, suppose w is indecomposible. Then, the utility function, u, is continuous and concave on Ω.*

PROOF. From Theorem 2.1, we know that u is concave and upper semicontinuous on Ω. So it remains to prove that u is lower semicontinuous on Ω as well.

Suppose, on the contrary, that u fails to be lower semicontinuous at some point $(x^o, z^o) \in \Omega$. Then, there is $\theta > 0$, and a sequence $(x^n, z^n) \to (x^o, z^o)$, such that

$$u(x^n, z^n) \leqslant u(x^o, z^o) - \theta \quad \text{for all } n. \tag{4.1}$$

Pick $y^o \in g(x^o, z^o)$ such that $w(y^o - z^o) = u(x^o, z^o)$. Since we clearly have $u(x^o, z^o) \geqslant w(z^o - z^o) = w(0)$, there are two cases to consider: (i) $w(y^o - z^o) = w(0)$; (ii) $w(y^o - z^o) > w(0)$.

In case (i), we pick $y^n = z^n$ for all n. Then $y^n \in g(x^n, z^n)$, and $u(x^n, z^n) \geqslant w(y^n - z^n) = w(0) = w(y^o - z^o) = u(x^o, z^o)$.This contradicts (4.1).

In case (ii), we know that $y^o \gg z^o$ since w is indecomposible. Pick $0 < \lambda < 1$, such that

$$\lambda y^o \gg z^o \quad \text{and} \quad [w(\lambda y^o - z^o) - w(y^o - z^o)] \geqslant (-\theta/4). \tag{4.2}$$

This is possible since w is continuous on R^m_+. Since $(x^n, z^n) \to (x^o, z^o)$, there is N such that for $n \geqslant N$

$$x^n \geqslant \lambda x^o,\ z^n \leqslant \lambda y^o, \quad \text{and} \quad w(\lambda y^o - z^n) \geqslant w(\lambda y^o - z^o) - (\theta/4). \tag{4.3}$$

Again this is possible since w is continuous on R^m_+. Since $(0, 0) \in \Omega$ and Ω is convex, so $(\lambda x^o, \lambda y^o) \in \Omega$ and by free-disposal (A.4), $(x^n, \lambda y^o) \in \Omega$ for $n \geqslant N$. Thus $\lambda y^o \in g(x^n, z^n)$ for $n \geqslant N$, and so $u(x^n, z^n) > w(\lambda y^o - z^n) = w(y^o - z^o) + [w(\lambda y^o - z^n) - w(\lambda y^o - z^o)] + [w(\lambda y^o - z^o) - w(y^o - z^o)] \geqslant w(y^o - z^o) - (\theta/2)$ (using (4.2) and (4.3)) $= u(x^o, z^o) - (\theta/2)$, which contradicts (4.1).

REMARKS.

(i) It will be obvious from the proofs of Theorem 2.1 and 4.2 that concavity of w is needed only to prove the concavity of u; thus, continuity of w implies continuity of u in Theorem 4.2 whether or not w is concave.

(ii) The proof of Theorem 4.1 uses both the concavity and upper semicontinuity of w to show the continuity of u. This makes the proof very short. We note, however, that using only (A.1) through (A.5) and the fact that Ω is locally simplicial, one can prove the lower hemicontinuity of g. This implies, by Theorem 2.1, that g is a continuous correspondence. Then, continuity of w implies the continuity of u (by the Maximum Theorem of Berge 1963), whether or not w is concave. (The proof is considerably longer, using this route.)

(iii) In the one-good model ($m = 1$), the welfare function is trivially indecomposible, so if w satisfies (A.6), (A.7), (or if w satisfies (A.8)) then by Theorem 4.2, u is continuous on Ω.

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